JOURNAL OF APPROXIMATION THEORY **85**, 69–87 (1996)
ARTICLE NO. 0029

Quantitative Approximation Theorems for Elliptic Operators*

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Communicated by Z. Ditzian

Received November 30, 1993; accepted in revised form April 4, 1995

Let $L(D)$ be an elliptic linear partial differential operator with constant coefficients and only highest order terms. For compact sets $K \subset \mathbf{R}^N$ whose complements are John domains we prove a quantitative Runge theorem: if a function f satisfies $L(D) f = 0$ on a fixed neighborhood of K, we estimate the sup-norm distance from f to the polynomial solutions of degree at most n. The proof utilizes a two-constants theorem for solutions to elliptic equations. We then deduce versions of Jackson and Bernstein theorems for elliptic operators.

1. INTRODUCTION AND STATEMENT OF RESULTS

In this paper we study quantitative approximation problems for solutions of elliptic partial differential equations. Throughout the paper we let $L(D)$ be an elliptic linear partial differential operator of order m on $\mathbf{R}^N$, with constant complex coefficients and only highest order terms. That is, we consider an operator $L(D) = \sum_{|\alpha| = m} a_\alpha D^\alpha$, where $L(x) \equiv \sum_{|\alpha| = m} a_\alpha x^\alpha$ is nonconstant polynomial with complex coefficients on $\mathbf{R}^N$ which is never

* Research supported in part by NSERC-Canada and AURC-New Zealand.

0021-9045/96 $18.00

equal to zero on $\mathbf{R}^N\backslash\{0\}$. We let $\mathscr{L}_n$ be the space of polynomials P of degree at most n satisfying $L(D)\,P\equiv 0$. If f is a continuous function on a compact set $K\subset\mathbf{R}^N$, we consider the distance

$$d_n(f, K)=\inf\{\|f-P\|_K\colon P\in\mathscr{L}_n\},$$

where we use the notation $\|g\|_s=\sup_s|g|$.

Lax and Malgrange have given an extension of the classical Runge approximation theorem to solutions of elliptic equations. Their result shows that if the compact set $K\subset\mathbf{R}^N$ has a connected complement, and f is a solution of $L(D)\,f=0$ on an open neighborhood of K, then $\lim_{n\to\infty} d_n(f, K)=0$. Our first theorem gives a quantitative version of this theorem, under certain regularity conditions, from which we will deduce Bernstein and Jackson theorems for solutions of elliptic equations.

A domain $\Omega\subset\mathbf{R}^N$ is called a **John domain** if $K=\mathbf{R}^N\backslash\Omega$ is a nonempty compact set, and there is a positive constant $J\leqslant 1$ with the following property: for each point $y\in\Omega$ there exists a locally rectifiable curve $\gamma(s)$ in Ω parameterized by arclength, with $\gamma(0)=y$ and $\gamma(\infty)=\infty$, such that $\operatorname{dist}(\gamma(s), K)\geqslant Js$ for every $s>0$. We refer to J as a **John constant** for Ω. If a regular subdomain G of class C^∞ in $\mathbf{R}^N\cup\{\infty\}$ contains the point ∞, then $G\backslash\{\infty\}$ is a John domain; this can be proved by making use of [GR, Chapter 2, Lemma 1.4 and the corollary of Lemma 1.7].

THEOREM 1 (Quantitative Runge Theorem). *Let K be a compact subset of $\mathbf{R}^N$ whose complement is a John domain. Then there are constants $p>0$, $b>1$, $q>0$, and $C>0$, each depending only on K and L, with the following property. If $0<\delta<1$, and f is a solution of $L(D)\,f=0$ on K_δ, then for any nonnegative integer $n\geqslant m-N$,*

$$d_n(f, K)\leqslant\frac{C}{\delta^p b^{n\delta^q}}\sup_{K_\delta}|f|. \tag{1}$$

Here and in the future we use the notation

$$X_\delta=\{x\colon \operatorname{dist}(x, X)<\delta\}$$

when $X\subset\mathbf{R}^N$ and $\delta>0$. We next state our Jackson theorem in the case when $m\leqslant N$ or N is odd.

THEOREM 2 (Elliptic Jackson Theorem). *Suppose $m\leqslant N$ or N is odd. Let K be a compact subset of $\mathbf{R}^N$ whose complement is a John domain. Then there are positive constants C_1 and C_2, depending only on K and L, with the following property. If f is a nonconstant continuous function on $\mathbf{R}^N$ with* *compact support which satisfies $L(D)f=0$ on the interior of K, then for any positive integer $n \geqslant m-N$,*

$$d_n(f, K) \leqslant C_1 \left(1+\frac{\|f\|_{\mathbf{R}^N}}{\omega(1)}\right) \omega\left(\frac{1}{n^{C_2}}\right), \tag{2}$$

where $\omega=\omega_f$ is the modulus of continuity

$$\omega(\delta)=\sup\{|f(x)-f(y)|: x\in\mathbf{R}^N, y\in\mathbf{R}^N, |x-y|<\delta\}.$$

We next state the Bernstein theorem for solutions of elliptic equations which was proved in [BL].

THEOREM 3 (Elliptic Bernstein Theorem [BL, Theorem 1.2]). *Let K be a nonempty compact subset of $\mathbf{R}^N$ with connected complement. Let Ω be an open neighborhood of K. Then there exists a constant $\rho<1$ such that for any solution f of $L(D)f=0$ on Ω we have* $\limsup_{n\to\infty} d_n(f, K)^{1/n} \leqslant \rho$.

Theorem 3 is easily deduced from Theorem 1. To see this, we note that the domain $\mathbf{R}^N \cup \{\infty\}\backslash K$ may be written as the union of an increasing sequence of regular subdomains of class C^∞. One of these subdomains must contain the compact set $\mathbf{R}^N \cup\{\infty\}\backslash\Omega$, and we let K' be the complement of this subdomain. We may then find $\delta>0$ so that $K'_\delta\subset\Omega$. Applying Theorem 1 to K', we obtain Theorem 3.

For the Cauchy–Riemann operator in $\mathbf{C}$, many Jackson-type results can be found in Dzyadyk [D], and Belyi [B] proved a precise Jackson theorem when K is the closure of a domain bounded by a quasiconformal curve. In the latter case the complement of K is a John domain; in fact, Andrievskii [A2] has characterized John domains in the plane in terms of a "k-quasidisk condition," and he has proved a precise Jackson theorem for harmonic functions in this two-dimensional setting. Andrievskii [A1] also used the John condition in proving an earlier version of our Theorem 2 for harmonic functions in $\mathbf{R}^N$.

A key ingredient in our proof of Theorem 1 is the following "two-constants" theorem.

THEOREM 4 (Two-Constants Theorem for Elliptic Operators). *Let Ω be a domain in $\mathbf{R}^N$. Let K be a compact subset of Ω, and G a nonempty open subset of Ω. Then there exist constants $Y\geqslant 1$ and $\tau\in(0,1)$, depending only on L, Ω, K, and G, with the following property. If f is any solution of $L(D)f=0$ in Ω, then*

$$\sup_K |f| \leqslant Y(\sup_G |f|)^\tau (\sup_\Omega |f|)^{1-\tau}. \tag{3}$$

We remark that the right side of (3) may be infinite, but cannot be of the indeterminate form $0 \cdot \infty$ because any solution of $L(D) f = 0$ is real-analytic. Theorem 4 remains valid for any elliptic partial differential operator with real-analytic is coefficients, and for a more general class of sets $G \subset \Omega$ as described at the end of Section 2. In case $L(D)$ is the Laplace operator, Korevaar and Meyers [KM] have proved Theorem 4 with $Y = 1$. For a discussion of related results and further references see [Kor, Section 5.1]. Vogt [V] has proved general theorems of this type using abstract functional analysis techniques.

In Section 2 we give some preliminary lemmas, including the "geometric" Lemma 1 which indicates the essential properties of John domains we will need; and we give the proof of Theorem 4. We prove Theorem 1 in Section 3, and in Section 4 we use Theorem 1 to prove Theorem 2. In the final Section 5 we give an extension of Theorem 2 to the case where $m > N$ and N is even.

2. Preliminary Results and the Proof of Theorem 4

If $a \in \mathbf{R}^N$ and $r > 0$, we use the notation $\mathbf{B}_r(a) = \{x \in \mathbf{R}^N : |x - a| < r\}$ and $\mathbf{A}_r(a) = \{x \in \mathbf{R}^N : |x - a| > r\}$, with the shortened forms $\mathbf{B}_r = \mathbf{B}_r(0)$ and $\mathbf{A}_r = \mathbf{A}_r(0)$. We will also have occasion to write $\mathbf{A}_{r, R} = \{x \in \mathbf{R}^N : r < |x| < R\}$.

We turn next to a discussion of John domains.

Lemma 1. *Let $\Omega \subset \mathbf{R}^N$ be a John domain, with John constant J, and let $K = \mathbf{R}^N \setminus \Omega$.*

(a) *Fix a radius $R > 1$ such that $K \subset \mathbf{B}_R$, and let $Q = 1 + J/8$. If $y \in \Omega \cap \mathbf{B}_R$ and $0 < \delta < 1$, then there is a sequence of points $a_0, a_1, ..., a_Z$ in Ω such that*

- (i) *the integer Z satisfies $Q^{Z-1} \leqslant 32R/(J\delta)$;*
- (ii) $|y - a_0| \leqslant \delta/8$;
- (iii) $|a_j - a_{j+1}| \leqslant J\,\delta Q^j/64$ *for* $0 \leqslant j \leqslant Z - 1$;
- (iv) $\operatorname{dist}(a_j, K) \geqslant J\,\delta Q^j/8$ *for* $0 \leqslant j \leqslant Z$;
- (v) $\mathbf{B}_{J\,\delta Q^j/16}(a_j) \subset \mathbf{B}_{8R/J}$ *for* $0 \leqslant j \leqslant Z$;
- (vi) $\mathbf{B}_{J\,\delta Q^Z/16}(a_Z) \subset \mathbf{A}_R$.

(b) *There is a constant $c > 0$ depending only on K with the following property. For any $\delta > 0$ and any $y \in \overline{K_{3\delta}} \cap \overline{\Omega_\delta}$, there exists a point $\tilde{y} \in \Omega$ with*

$$\operatorname{dist}(\tilde{y}, K) \geqslant 4\delta, \tag{4}$$

$$|y - \tilde{y}| \leqslant c\delta. \tag{5}$$

If $\delta > 0$ is fixed, we can arrange that the map $y \to \tilde{y}$ assumes only finitely many values, and the inverse image of each value is a Borel set.

Proof. (a) Let γ be the arc associated with the point $y \in \Omega$ in the definition of a John domain, and select the positive integer Z so that

$$\log_Q \frac{32R}{J\delta} \leqslant Z < \left(\log_Q \frac{32R}{J\delta}\right) + 1.$$

Let $a_j = \gamma(\delta Q^j/8)$ for $j \in \{0, 1, ..., Z\}$. Then property (i) is clear, property (ii) follows from writing

$$|y - a_0| = |\gamma(0) - \gamma(\delta/8)| \leqslant \delta/8,$$

and property (iii) from writing

$$|a_j - a_{j+1}| = |\gamma(\delta Q^j/8) - \gamma(\delta Q^{j+1}/8)| \leqslant \delta Q^j(Q-1)/8 = J\,\delta Q^j/64.$$

Property (iv) follows from the definition of John domain, and property (v) from the estimate

$$|a_j| \leqslant |a_j - y| + |y| = |\gamma(\delta Q^j/8) - \gamma(0)| + |y| \leqslant \frac{\delta Q^j}{8} + R, \qquad \text{for} \quad 0 \leqslant j \leqslant Z$$

and property (i). Property (vi) follows from noting that

$$\operatorname{dist}(a_Z, K) = \operatorname{dist}(\gamma(\delta Q^Z/8), K) \geqslant \frac{J\delta Q^Z}{8} \geqslant \frac{J\delta Q^Z}{16} + 2R.$$

(b) For each point $y \in \overline{K_{3\delta}} \cap \overline{\Omega_\delta}$ there is a point $z_y \in \Omega$ such that $|z_y - y| < 2\,\delta$. By the John property, there is a point $\tilde{y} \in \Omega$ such that

$$\operatorname{dist}(\tilde{y}, K) \geqslant 4\delta$$

and

$$|\tilde{y} - z_y| \leqslant 4\,\delta/J,$$

so the point $\tilde{y}$ satisfies both (4) and (5). Using the compactness of $\overline{K_{3\delta}} \cap \overline{\Omega_\delta}$, we can arrange that each of the maps $y \to z_y$ and $y \to \tilde{y}$ assumes only finitely many values, and the inverse image of each value is a Borel set.

We recall that a distribution E on $\mathbf{R}^N$ is called a *fundamental solution* for $L(D)$ if $L(D)\,E$ is equal to the unit measure supported at the origin. The following lemma establishes the existence of fundamental solutions for the operators considered in this paper ([J, Chapter 3], [H, Chapter 7]). We let $\mathscr{P}_l$ denote the space of polynomials in N real variables, with complex coefficients, which are homogeneous of degree l.

LEMMA 2. *There exists a fundamental solution for* $L(D)$ *which is a locally integrable function on* $\mathbf{R}^N$ *of the form* $E(x) \equiv E_1(x) + E_2(x) \log |x|$, *where the restriction of* E_1 *to* $\mathbf{R}^N \setminus \{0\}$ *is real-analytic and homogeneous of degree* $m - N$, *and*

$$E_2 = 0 \qquad \text{if} \quad m < N \quad \text{or } N \text{ is odd};$$

$$E_2 \in \mathscr{P}_{m-N} \qquad \text{if} \quad m \geqslant N \quad \text{and } N \text{ is even}.$$

We now summarize some well-known facts concerning the fundamental solution E; we refer to [BL] for further discussion and references. If $x \in \mathbf{R}^N \setminus \{0\}$ is fixed, the function $y \to E(x-y)$ is real-analytic on $\mathbf{R}^N \setminus \{0\}$; we may write the Taylor series expansion in y about 0,

$$E(x-y) = \sum_{l=0}^{\infty} Q_l^{(x)}(y), \tag{6}$$

where

$$Q_l^{(x)}(y) = (-1)^l \sum_{|\alpha| = l} \frac{D^\alpha E(x)}{\alpha!} y^\alpha. \tag{7}$$

This expansion is valid or y in some neighborhood of the origin in $\mathbf{R}^N$. It follows that for fixed $x \in \mathbf{R}^N \setminus \{0\}$, each polynomial $Q_l^{(x)}$ satisfies

$$L(D)\, Q_l^{(x)} \equiv 0 \qquad \text{on } \mathbf{R}^N.$$

LEMMA 3. *There is a constant* $A > 1$ *with he following property. If* n *is a nonnegative integer satisfying* $n \geqslant m - N$, *and* μ *is a complex measure on* $\bar{\mathbf{B}}_r$ *satisfying*

$$\int P \, d\mu = 0 \qquad \text{for all} \quad P \in \mathscr{L}_n,$$

then

$$|E * \mu(x)| \leqslant \left(\frac{|x|}{r}\right)^{m-N} \left(\frac{Ar}{|x|}\right)^{n+1} \sup_{\mathbf{A}_{r,\,2r}} |E * \mu| \qquad \text{for} \quad |x| \geqslant Ar.$$

Lemma 3 follows from [BL, Theorem 4.1, Theorem 4.2, Remark 4.3, and Theorem 5.2]. *For the rest of this paper we let* A *denote the constant of Lemma* 3.

LEMMA 4. *For each multi-index* α *there is a constant* $C(\alpha)>0$ *with the following property. If* f *satisfies* $L(D)f=0$ *on an open ball* $\mathbf{B}_\rho(a)$, *where* $a\in\mathbf{R}^N$ *and* $\rho>0$, *then*

$$|D^\alpha f(a)|\leqslant\frac{C(\alpha)}{\rho^{|\alpha|}}\sup_{\mathbf{B}_\rho(a)}|f|.$$

Lemma 4 follows from applying [BL, Theorem 5.3] to the function $u(x)\equiv f(a+\rho x)$ on the unit ball $\mathbf{B}_1$.

We close this section with the proof of Theorem 4, for which we need the following result.

LEMMA 5. *Let* $\tilde{\Omega}$ *be a bounded domain in* $\mathbf{C}^N$. *Let* $G\subset\tilde{\Omega}\cap\mathbf{R}^N$ *be a non-empty open subset of* $\mathbf{R}^N$, *and* K *a compact subset of* $\tilde{\Omega}$. *Then there exists a constant* $\tau\in(0,1)$, *depending only on* $\tilde{\Omega}$, G, *and* K, *with the following property. If* g *is a holomorphic function on* $\tilde{\Omega}$ *which satisfies* $|g|\leqslant M$ *on* $\tilde{\Omega}$, *and if* $|g|\leqslant m\leqslant M$ *on* G, *then*

$$|g|\leqslant m^\tau M^{1-\tau}\qquad \text{on } K.$$

Lemma 5 follows from the two-constants lemma for *plurisubharmonic* functions in [Kl, Proposition 4.5.6]; see the remarks following [BL, Lemma 3.1]. (Actually, this argument might give Lemma 5 with the constant $\tau=1$, but then Lemma 5 holds *a fortiori* with the constant $\tau=1/2$.)

Proof of Theorem 4. This proof should be compared with arguments in [BL, Section 3]. For the proof we may assume that $G\subset\subset\Omega$. The domain Ω may be written as the union of an increasing sequence of relatively compact subdomains; one of these subdomains must contain the compact set $K\cup\bar{G}$, and we let Ω' be a subdomain with this property.

We will use the fact that for each positive number R there exist positive numbers $r(R)<R$ and $C(R)$ with the following property [ABG, Lemma 2]; *if* h *is any solution of* $L(D)\,h=0$ *on a ball* $\mathbf{B}_R(a)\subset\mathbf{R}^N$, *then there is a (unique) holomorphic function* $\tilde{h}$ *on the complex ball* $\tilde{\mathbf{B}}_{r(R)}(a)\equiv\{z\in\mathbf{C}^N\colon|z-a|<r(R)\}$ *which agrees with* h *on the real ball* $\mathbf{B}_{r(R)}(a)$, *and* $\|\tilde{h}\|_{\tilde{\mathbf{B}}_{r(R)}(a)}\leqslant C(R)\,\|h\|_{\mathbf{B}_R(a)}$. Maintaining this notation, we note that each point $a\in\overline{\Omega'}$ is the center of an open ball $\mathbf{B}_{R(a)}(a)\subset\Omega$, and by the Heine-Borel theorem we can find a finite set $\mathscr{F}\subset\overline{\Omega'}$ such that

$$\overline{\Omega'}\subset\bigcup_{a\in\mathscr{F}}\mathbf{B}_{r(R(a))}(a).$$

Thus the union

$$\tilde{\Omega}\equiv\bigcup_{a\in\mathscr{F}}\tilde{\mathbf{B}}_{r(R(a))}(a)$$

is an open set in $\mathbf{C}^N$ containing $\overline{\Omega'}$, and $\tilde{\Omega}$ is connected since it can be regarded as the union of the connected set $\overline{\Omega'}$ and balls $\tilde{\mathbf{B}}_{r(R(a))}(a)$ which intersect $\overline{\Omega'}$.

To complete the proof of Theorem 4, it suffices to prove (3) when f is any solution of $L(D) f=0$ in Ω satisfying $\sup_\Omega |f| < \infty$. From the italicized result above we know that for each point $a \in \mathscr{F}$ there is a holomorphic function $\tilde{f}_a$ on the complex ball $\tilde{\mathbf{B}}_{r(R(a))}(a)$ which agrees with f on the real ball $\mathbf{B}_{r(R(a))}(a)$. We now obtain a well-defined holomorphic function g on $\tilde{\Omega}$ by requiring that $g=\tilde{f}_a$ on $\tilde{\mathbf{B}}_{r(R(a))}(a)$; in particular, $g \equiv f$ on $\tilde{\Omega} \cap \mathbf{R}^N \supset \overline{\Omega'}$. Moreover, the italicized result above shows that

$$\sup_{\tilde{\Omega}} |g| \leqslant \tilde{C} \sup_{\Omega} |f|,$$

where $\tilde{C}=\sup_{a \in \mathscr{F}} C(R(a))$. We now see from Lemma 5 that there is a constant $\tau \in (0, 1)$, depending only on L, K, and G, such that

$$\sup_K |f| \leqslant (\sup_G |f|)^\tau \, (\sup_{\tilde{\Omega}} |g|)^{1-\tau} \leqslant \tilde{C}^{1-\tau} (\sup_G |f|)^\tau \, (\sup_\Omega |f|)^{1-\tau},$$

so Theorem 4 holds with $Y=\tilde{C}^{1-\tau}$.

Remark. Theorem 4 is valid if $L(D)$ is any elliptic partial differential operator with real-analytic coefficients as we see from a similar argument using [G, Lemma, p. 153] instead of [ABG, Lemma 2]. In addition, we may replace the condition that $G \subset \Omega$ be a nonempty open set by the less restrictive hypothesis that $G \subset \mathbf{R}^N \subset \mathbf{C}^N$ be *nonpluripolar* since Lemma 5 remains valid [K, Chapter 4].

3. Proof of Theorem 1

We begin with the following corollary of Theorem 4.

Lemma 6. *There exist constants $Y \geqslant 1$ and $\tau \in (0, 1)$ with the following property. Let B be an open ball of radius ρ in $\mathbf{R}^N$, and let a and $\tilde{a}$ be points in the ball B whose distance from the center is no more than $\rho/4$. If f is any solution of $L(D) f=0$ in B, then*

$$\sup_{\mathbf{B}_{\rho/8}(\tilde{a})} |f| \leqslant Y(\sup_{\mathbf{B}_{J_\rho/64}(a)} |f|)^\tau \, (\sup_B |f|)^{1-\tau}.$$

Proof. From Theorem 4 we see that there are constants $Y \geqslant 1$ and $\tau \in (0, 1)$ such that any solution u of $L(D)\, u=0$ in $\mathbf{B}_{3/4}$ satisfies

$$\sup_{\mathbf{B}_{5/8}} |u| \leqslant Y(\sup_{\mathbf{B}_{J/64}} |u|)^\tau \, (\sup_{\mathbf{B}_{3/4}} |u|)^{1-\tau}. \tag{8}$$

Under the hypotheses of Lemma 6 we then have

$$\sup_{\mathbf{B}_{\rho/8}(\tilde{a})} |f| \leqslant \sup_{\mathbf{B}_{5\rho/8}(a)} |f| \leqslant Y(\sup_{\mathbf{B}_{J\rho/64}(a)} |f|)^{\tau} (\sup_{\mathbf{B}_{3\rho/4}(a)} |f|)^{1-\tau}$$
$$\leqslant Y(\sup_{\mathbf{B}_{J\rho/64}(a)} |f|)^{\tau} (\sup_{B} |f|)^{1-\tau}.$$

Here the first inequality follows from the inclusion $\mathbf{B}_{\rho/8}(\tilde{a}) \subset \mathbf{B}_{5\rho/8}(a)$; and the second and third inequalities follow from noting that $\mathbf{B}_{3\rho4}(a) \subset B$ and applying (8) to the function $u(x) \equiv f(a + \rho x)$. This proves Lemma 6.

LEMMA 7. *Let K be a nonempty compact subset of $\mathbf{R}^N$ whose complement is a John domain with John constant J, and let r be the smallest positive number such that $K_1 \subset \mathbf{B}_r$. Then there are constants $b > 1$, $q > 0$, and $C > 0$, each depending only on K and L, with the following property. If n is a nonnegative integer satisfying $n \geqslant m - N$, μ is a complex measure on K satisfying*

$$\int P\, d\mu = 0 \qquad \textit{for all} \quad P \in \mathscr{L}_n,$$

and $0 < \delta < 1$, *then*

$$\sup_{\mathbf{B}_{2Ar} \setminus K_\delta} |E * \mu| \leqslant \frac{C}{b^{n\,\delta^q}} \sup_{\mathbf{B}_{16Ar/J} \setminus K_{J\,\delta/16}} |E * \mu|. \tag{9}$$

Proof. We define $u = E * \mu$, and let $\delta \in (0, 1)$ be fixed. Without loss of generality, we will prove (9) under the additional assumption that

$$\sup_{\mathbf{B}_{16Ar/J} \setminus K_{J\,\delta/16}} |u| = 1. \tag{10}$$

Since $J\,\delta/16 < 1$, it then follows from Lemma 3 that

$$|u(x)| \leqslant \left(\frac{|x|}{r}\right)^{m-N} \left(\frac{Ar}{|x|}\right)^{n+1} \qquad \text{for} \quad |x| \geqslant Ar. \tag{11}$$

For the rest of the proof of Lemma 7, we fix a point $y \in \mathbf{B}_{2Ar} \setminus K_\delta$, and we will prove that $|E * \mu(y)|$ is bounded by the right side of (9). We let $a_0, a_1, ..., a_Z$ be a sequence of points associated with the point y and the radius $R = 2Ar$ in Lemma 1(a). Then the ball $\mathbf{B}_{J\delta Q^Z/128}(a_Z)$ is contained in $\mathbf{A}_{R,\,8R/J}$, and in particular we may apply (11) to all points x in this ball; we conclude that

$$\sup_{x \in \mathbf{B}_{J\delta Q^Z/128}(a_Z)} |u(x)| \leqslant \frac{r^{N-m}}{2^{n+1}} \sup_{x \in \mathbf{B}_{J\delta Q^Z/128}(a_Z)} (|x|^{m-N})$$
$$\leqslant \frac{r^{N-m}}{2^{n+1}} \sup_{x \in \mathbf{A}_{R,\,8R/J}} (|x|^{m-N}) \equiv \frac{C_1}{2^n}. \tag{12}$$

Here the symbol $\equiv$ indicates that we are defining the constant $C_1 = r^{N-m} 2^{-1} \sup_{x \in \mathbf{A}_{R,\, 8R/J}} (|x|^{m-N})$. We want to use estimate (12), in conjunction with repeated applications of Lemma 6, to estimate $|u(y)|$.

From Lemma 1(a) we may verify that the balls $B_j \equiv \mathbf{B}_{J\delta Q^j/16}(a_j)$, $j = 0, ..., Z-1$ are all contained in the set $\mathbf{B}_{8R/J} \backslash K_{J\delta/16}$, and hence from (10) we have $\sup_{B_j} |u| \leqslant 1$. Applying Lemma 6 to the ball B_{Z-1} gives

$$\sup_{\mathbf{B}_{J\delta Q^{Z-1}/128}(a_{Z-1})} |u| \leqslant Y(\sup_{\mathbf{B}_{J\delta Q^{Z}/128}(a_Z)} |u|)^\tau \leqslant Y\left(\frac{C_1}{2^n}\right)^\tau. \tag{13}$$

Here the second inequality follows from (12). Next, we have

$$\sup_{\mathbf{B}_{J\delta Q^{Z-2}/128}(a_{Z-2})} |u| \leqslant Y(\sup_{\mathbf{B}_{J\delta Q^{Z-1}/128}(a_{Z-1})} |u|)^\tau \leqslant Y^{1+\tau} \left(\frac{C_1}{2^n}\right)^{\tau^2}$$

where the first inequality follows from applying Lemma 6 to the ball B_{Z-2}, and the second inequality follows from (13). Continuing inductively, we conclude that

$$\sup_{\mathbf{B}_{J\delta Q^{0}/128}(a_0)} |u| \leqslant Y^{1+\tau+\cdots+\tau^{Z-1}} \left(\frac{C_1}{2^n}\right)^{\tau^Z} \leqslant Y^{1/1-\tau} \frac{\max\{1, C_1\}}{2^{n\tau^Z}}.$$

Finally we note that the ball $\mathbf{B}_{\delta/2}(y)$ is contained in the set $\mathbf{B}_{8R/J} \backslash K_{J\delta/16}$, and in particular $|u| \leqslant 1$ on this ball; from this and property (ii) of Lemma 1(a) we see that we may apply Lemma 6 to this ball to obtain the estimate

$$|u(y)| \leqslant Y(\sup_{\mathbf{B}_{J\delta/128}(a_0)} |u|)^\tau \leqslant Y^{1/1-\tau} \frac{(\max\{1, C_1\})^\tau}{2^{n\tau^{Z+1}}}.$$

Now using property (i) in Lemma 1(a) gives Lemma 7.

We now give the proof of Theorem 1, which is a refinement of the proof of the Bernstein theorem in [BL, Section 3]. We let r be the smallest positive number such that $K_1 \subset \mathbf{B}_r$. We let $\phi \in C_0^\infty(\mathbf{B}_1) \subset C_0^\infty(\mathbf{R}^N)$ be a fixed nonnegative function with $\int \phi(x)\, dx = 1$, and for each $\rho > 0$ we let

$$\phi_\rho(x) \equiv \frac{1}{\rho^N} \phi\left(\frac{x}{\rho}\right).$$

If we define

$$N_l = \sup\{|D^\alpha \phi(x)| : x \in \mathbf{R}^N, |\alpha| = l\},$$

for each nonnegative integer l, then

$$|D^\alpha \phi_\rho(x)| \leqslant \frac{N_{|\alpha|}}{\rho^{N+|\alpha|}}. \tag{14}$$

Now let $\delta \in (0, 1)$. We define

$$\psi = \chi_{K_{3\delta/4}} * \phi_{\delta/8},$$

where $\chi_{K_{3\delta/4}}$ denotes the characteristic function of the set $K_{3\delta/4}$, and we note that the function $\psi \in C_0^\infty(K_{7\delta/8})$ is identically equal to one on $K_{5\delta/8}$. Using (14) we see that

$$|D^\alpha \psi(x)| \leqslant N_{|\alpha|} \left(\frac{8}{\delta}\right)^{N+|\alpha|}. \tag{15}$$

For each fixed nonnegative integer $n \geqslant m - N$, we may apply the Hahn–Banach theorem and the Riesz representation theorem to see that there is a complex Borel measure $\mu = \mu_n$ of total variation one supported on K such that

$$\int P \, d\mu = 0 \qquad \text{for all} \quad P \in \mathscr{L}_n,$$

and

$$d_n = d_n(f, K) = \int f \, d\mu.$$

We may regard $F \equiv \psi f \in C_0^\infty(K_\delta) \subset C_0^\infty(\mathbf{R}^N)$, and then

$$d_n = \int_K F \, d\mu = (-1)^m \int_{K_\delta \setminus K_{\delta/2}} (E * \mu)(x) \, L(D) \, F(x) \, dx. \tag{16}$$

(See [BL, Section 3].) We wish to estimate the functions appearing in the last integrand in (16).

From Lemma 7 we see that there are constants $b > 1$, $q > 0$, $C > 0$ such that

$$\sup_{K_\delta \setminus K_{\delta/2}} |E * \mu| \leqslant \frac{C}{b^{n\delta^q}} \sup_{\mathbf{B}_{16Ar/J} \setminus K_{J\delta/32}} |E * \mu|. \tag{17}$$

Now from the form of the fundamental solution E given in Lemma 2, and the fact that $\int |d\mu| \leqslant 1$, we see that there is a constant $\tilde{C} > 0$ such that

$$\sup_{\mathbf{B}_{16Ar/J} \setminus K_{J\delta/32}} |E * \mu| \leqslant \sup\{|E(x-y)| : x \in \mathbf{B}_{16Ar/J} \setminus K_{J\delta/32}, \, y \in K\}$$

$$\leqslant \begin{cases} \tilde{C} & \text{if} \quad m > N \\ \tilde{C}(1 + |\log \delta|) & \text{if} \quad m = N \\ \tilde{C}\delta^{m-N} & \text{if} \quad m < N. \end{cases} \tag{18}$$

Finally, there are complex constants $c_{\alpha\beta}$, depending only on L, such that

$$L(D)\,F(x) = L(D)(\psi f)(x) = \sum_{|\alpha+\beta|=m} c_{\alpha\beta}(D^\alpha \psi(x))(D^\beta f(x)). \tag{19}$$

It is clear that supp $L(D)\,F \subset K_{7\delta/8} \setminus K_{5\delta/8}$, so each point $x \in \operatorname{supp} L(D)\,F$ is the center of an open ball $\mathbf{B}_{\delta/8}(x)$ in K_δ; we then conclude from Lemma 4 that

$$|D^\beta f(x)| \leqslant \frac{M}{\delta^{|\beta|}} \sup_{\mathbf{B}_{\delta/8}(x)} |f| \leqslant \frac{M}{\delta^{|\beta|}} \sup_{K_\delta} |f|$$

$$\text{for} \quad x \in \operatorname{supp} L(D)\,F \quad \text{and} \quad |\beta| \leqslant m, \tag{20}$$

where $M = \max\{8^{|\gamma|} C(\gamma) : |\gamma| \leqslant m\}$, and $C(\gamma)$ indicates the constant in Lemma 4.

Theorem 1 now follows from substituting equation (19) into equation (16), and then using the estimates in (15), (17), (18), and (20).

4. Proof of Theorem 2

For the rest of the paper, we let f be a nonconstant continuous function on $\mathbf{R}^N$ with compact support which satisfies $L(D)\,f = 0$ on the interior of K. The letter k will often be used to denote any constant which can depend only on K, L and N. For $\delta > 0$, we take the convolution

$$g(x) \equiv (f * \phi_\delta)(x) \tag{21}$$

with ϕ_δ as in Section 3, and we make the following three observations.

(i) $\|g - f\|_{\mathbf{R}^N} \leqslant \omega(\delta)$;

(ii) $\|L(D)\,g\|_{\mathbf{R}^N} \leqslant k\omega(\delta)\,\delta^{-m}$;

(iii) $L(D)\,g = 0$ outside Ω_δ.

Here (i) follows from (21); (ii) follows from noting that for each $x_0 \in \mathbf{R}^N$ we have

$$[L(D)\,g](x_0) = [f * L(D)\,\phi_\delta](x_0) = [(f - f(x_0)) * L(D)\,\phi_\delta](x_0),$$

and using (14); and (iii) follows from the fact that $L(D)\,f = 0$ on the interior of K.

We have

$$g(x) = \int_{\Omega_\delta} E(x-y)\,L(D)\,g(y)\,dy$$

for all x. Our next step is to modify g to get a function $u \in C^\infty(\mathbf{R}^N)$ satisfying

(a) $\|u-g\|_{K_{3\delta}} \leqslant k\omega(\delta)$;

(b) $\|u\|_{K_{3\delta}} \leqslant k\omega(\delta) + \|f\|_{\mathbf{R}^N}$;

(c) $L(D)\, u = 0$ on the neighborhood $K_{3\delta}$ of K.

Then we will be ready to apply the Theorem 1 to u. We need to approximate the fundamental solution, $E(x-y) = E((x-\tilde{y})-(y-\tilde{y}))$.

Definition. For fixed $y \in K_{3\delta} \cap \Omega_\delta$ we define

$$\psi_y(x) = \sum_{l=0}^{m} Q_l^{(x-\tilde{y})}(y-\tilde{y}),$$

where $\tilde{y} \in \Omega$ is chosen as in Lemma 1(b).

From the definition of Q in (7) we see that $L(D)\, \psi_y = 0$ on $\mathbf{R}^N \backslash \{\tilde{y}\}$. In particular, from (4), $L(D)\, \psi_y = 0$ on $K_{3\delta}$. Therefore, the function

$$u(x) \equiv \int_{\Omega_\delta \backslash K_{3\delta}} E(x-y)\, L(D)\, g(y)\, dy + \int_{\Omega_\delta \cap K_{3\delta}} \psi_y(x)\, L(D)\, g(y)\, dy$$

satisfies (c).

To verify (a), note that

$$u(x) - g(x) = \int_{\Omega_\delta \cap K_{3\delta}} L(D)\, g(y)[E(x-y) - \psi_y(x)]\, dy.$$

For each $y \in \Omega_\delta \cap K_{3\delta}$, we need to estimate $|E(x-y) - \psi_y(x)|$ for $x \in K_{3\delta}$ *away* from y and we also need to bound the integrals of $|E(x-y)|$ and $|\psi_y(x)|$ when x is *near* y. The quantitative estimates we need are in Lemma 9; we first recall the following result from [BL].

Lemma 8 ([BL, Lemma 2.2 and Corollary 2.3]). *There exists a constant $M_0 > 1$ with the following property. If α is a multiindex, and we assume that $|\alpha| > m-N$ in case N is even, then*

$$|D^\alpha E(x)| \leqslant \alpha!\, M_0^{|\alpha|}\, |x|^{m-N-|\alpha|}, \qquad x \in \mathbf{R}^N \backslash \{0\}.$$

In particular, if l is a nonnegative integer and we assume that $l > m-N$ in case N is even,

$$|Q_l^{(x)}(y)| \leqslant |x|^{m-N-l}\, (M_0\, |y|)^l \sum_{|\alpha|=l} 1 \qquad \text{if} \quad x \in \mathbf{R}^N \backslash \{0\} \quad \text{and} \quad y \in \mathbf{R}^N.$$

Furthermore, if $|x| > M_0\, |y|$, then (6) holds.

LEMMA 9. *Assume that $m < N$ or N is odd. There exist positive constants c_1, c_2, c_3 depending only on K and L such that for any $\delta \in (0, 1)$ and any $y \in \Omega_\delta \cap K_{3\delta}$,*

$$|\psi_y(x)| \leqslant c_1 \, |x - \tilde{y}|^{m-N} \qquad \text{for} \quad x \in K_{3\delta}, \tag{22}$$

$$|E(x-y) - \psi_y(x)| \leqslant \frac{c_2 \delta^{m+1}}{|x-y|^{N+1}} \qquad \text{for} \quad |x-y| \geqslant c_3 \delta. \tag{23}$$

Equation (23) *remains valid if $m \geqslant N$ and N is even.*

Proof. If $x \in K_{3\delta}$, then we see from (4) that $|x - \tilde{y}| \geqslant \delta$; using this fact, (5), and Lemma 8, we obtain (22).

We will now prove (23) for any $c_3 > 0$ satisfying

$$c_3 > c(2M_0 + 1) > c$$

where c is the constant in (5). Then

$$\frac{2M_0 c}{(1 - (c/c_3)) \, c_3} < 1. \tag{24}$$

Fix x with $|x - y| \geqslant c_3 \delta$. Thus $\delta \leqslant |x-y|/c_3$ and, from (5),

$$|y - \tilde{y}| < c\delta \leqslant \frac{c}{c_3} \, |x - y|.$$

Hence

$$|x - \tilde{y}| \geqslant |x - y| - |y - \tilde{y}| \geqslant \left(1 - \frac{c}{c_3}\right) |x - y|. \tag{25}$$

From (24) and (25) it follows that $|x - \tilde{y}| > 2M_0 \, |y - \tilde{y}|$, so that by Lemma 8 we have

$$E(x-y) = E((x-\tilde{y}) - (y - \tilde{y})) = \sum_{l=0}^{\infty} Q_l^{(x-\tilde{y})}(y - \tilde{y})$$

for $|x-y| \geqslant c_3 \delta$. Applying the estimate in Lemma 8 for $l > m > m - N$, and using (5) and (25), we obtain

$$\begin{aligned} |E(x-y) - \psi_y(x)| &= \left| \sum_{l=m+1}^{\infty} Q_l^{(x-\tilde{y})}(y-\tilde{y}) \right| \\ &\leqslant \sum_{l=m+1}^{\infty} \frac{(M_0 c\delta)^l}{\{(1 - (c/c_3)) \, |x-y|\}^{l-m+N}} \sum_{|\alpha| = l} 1. \end{aligned}$$

Since $|x-y| \geqslant c_3\delta$, by (24) it follows that $M_0 c\delta/((1-(c/c_3))\,|x-y|) < 1/2$. Thus (23) follows from the elementary estimate

$$\sum_{l=m+1}^{\infty} r^l \sum_{|\alpha|=l} 1 \leqslant r^{m+1} 2^{m+1} \sum_{|\alpha| \geqslant 0} \left(\tfrac{1}{2}\right)^{|\alpha|} = r^{m+1} 2^{N+m+1}, \qquad 0 < r < 1/2.$$

This completes the proof of Lemma 9.

Recall that

$$g(x) = \int_{\Omega_\delta} E(x-y)\, L(D)\, g(y)\, dy$$

and

$$u(x) \equiv \int_{\Omega_\delta \setminus K_{3\delta}} E(x-y)\, L(D)\, g(y)\, dy + \int_{\Omega_\delta \cap K_{3\delta}} \psi_y(x)\, L(D)\, g(y)\, dy$$

where ψ_y are the functions from Lemma 9.

We want to estimate $\|u-g\|_{K_{3\delta}}$. Fix $x \in K_{3\delta}$. Then

$$|u(x)-g(x)| \leqslant \int_{\Omega_\delta \cap K_{3\delta}} |L(D)\, g(y)|\; |E(x-y) - \psi_y(x)|\; dy = I_1 + I_2$$

where

$$I_1 \equiv \int_{\{|x-y| \leqslant c_3\delta\} \cap \Omega_\delta \cap K_{3\delta}} |L(D)\, g(y)|\; |E(x-y) - \psi_y(x)|\; dy \tag{26}$$

$$I_2 \equiv \int_{\{|x-y| > c_3\delta\} \cap \Omega_\delta \cap K_{3\delta}} |L(D)\, g(y)|\; |E(x-y) - \psi_y(x)|\; dy. \tag{27}$$

We first estimate I_2. Since the integration takes place over $\{y\colon |x-y| > c_3\delta\}$, by (23) and (ii) follows that

$$I_2 \leqslant \frac{k\omega(\delta)}{\delta^m} \int_{\{|x-y| > c_3\delta\}} c_2\delta^{m+1} \frac{1}{|x-y|^{N+1}}\, dy = \frac{k\omega(\delta)}{\delta^m}\, c_2\delta^{m+1} \frac{1}{c_3\delta}$$

which proves that $I_2 \leqslant k\omega(\delta)$.

To estimate I_1, first assume $m < N$ or N is odd and use the estimate (ii) to obtain

$$\begin{aligned} I_1 &\leqslant \frac{k\omega(\delta)}{\delta^m} \left[\int_{\mathbf{B}_{c_3\delta}(x)} |E(x-y)|\, dy + \int_{\mathbf{B}_{c_3\delta}(x) \cap K_{3\delta}} |\psi_y(x)|\, dy \right] \\ &\leqslant \frac{k\omega(\delta)}{\delta^m} \left[\int_{\{|z| < c_3\delta\}} |E(z)|\, dz + k\delta^m \right] \leqslant k\omega(\delta), \end{aligned}$$

where the second inequality follows from (22), the inequality

$$\delta \leqslant |x-\tilde{y}| \leqslant (c_3+c)\,\delta, \qquad x \in K_{3\delta}, \tag{28}$$

and the form of the fundamental solution if $m<N$ or N is odd. This gives (a) if $m<N$ or N is odd.

To estimate I_1 when $m=N$ and N is even, note that $Q_0^{(x-\tilde{y})}(y-\tilde{y})=E(x-\tilde{y})$ and hence

$$|E(x-y)-\psi_y(x)| = \left|E(x-y)-E(x-\tilde{y})-\sum_{l=1}^{m} Q_l^{(x-\tilde{y})}(y-\tilde{y})\right|.$$

We now estimate

$$I_3 \equiv \int_{\mathbf{B}_{c_3\delta}(x)\,\cap\,\Omega_\delta\,\cap\,K_{3\delta}} |E(x-y)-E(x-\tilde{y})|\,dy$$

and

$$I_4 \equiv \int_{\mathbf{B}_{c_3\delta}(x)} \sum_{l=1}^{m} |Q_l^{(x-\tilde{y})}(y-\tilde{y})|\,dy.$$

For the latter integral, since $l>0=m-N$, from Lemma 8 we obtain

$$|Q_l^{(x-\tilde{y})}(y-\tilde{y})| \leqslant \left(\frac{cM_0}{c_3}\right)^l \sum_{|\alpha|=l} 1,$$

so that $I_4 \leqslant k\,\delta^m$ since $\mathrm{vol}(\mathbf{B}_{c_3\delta}(x))=k\,\delta^N$. For the estimate on I_3 we write $E(x)=E_1(x)+E_2\log|x|$ where E_2 is a constant and E_1 is homogeneous of degree 0 away from the origin. Thus

$$\int_{\mathbf{B}_{c_3\delta}(x)} |E_1(x-y)-E_1(x-\tilde{y})|\,dy \leqslant k\delta^N = k\delta^m.$$

To estimate

$$\int_{\mathbf{B}_{c_3\delta}(x)\,\cap\,\Omega_\delta\,\cap\,K_{3\delta}} \left|\log\frac{|x-y|}{|x-\tilde{y}|}\right|\,dy,$$

from (28) we have

$$\frac{|x-y|}{(c+c_3)\,\delta} \leqslant \frac{|x-y|}{|x-\tilde{y}|} \leqslant \frac{|x-y|}{\delta}$$

for $y \in \mathbf{B}_{c_3\delta}(x) \cap \Omega_\delta \cap K_{3\delta}$. Thus

$$\left|\log \frac{|x-y|}{|x-\tilde{y}|}\right| \leqslant \left|\log \frac{|x-y|}{\delta}\right| + \left|\log \frac{|x-y|}{(c+c_3)\,\delta}\right|.$$

An elementary computation shows that

$$\int_0^{c_3\delta} \left|\log\left(\frac{r}{c\delta}\right)\right| r^{N-1}\,dr \leqslant k\delta^N.$$

It follows that $I_3 \leqslant k\,\delta^m$ and hence

$$\int_{\mathbf{B}_{c_3\delta}(x)\cap\Omega_\delta\cap K_{3\delta}} |E(x-y)-\psi_y(x)|\,dy \leqslant k\delta^m$$

from which it follows that $I_1 \leqslant k\omega(\delta)$. This yields (a) when $m=N$ and N is even.

We then obtain (b) using (a) and the triangle inequality:

$$|u(x)| \leqslant |u(x)-g(x)| + |g(x)| \leqslant k\omega(\delta) + \|f\|_{\mathbf{R}^N} \qquad \text{for} \quad x \in K_{3\delta}.$$

We can now complete the proof of Theorem 2. Given the function f and $0<\delta<1$, we construct the function u satisfying (a), (b), and (c). Applying Theorem 1 to u on K_δ, we obtain

$$d_n(u, K) \leqslant C\,\frac{k\omega(\delta) + \|f\|_{\mathbf{R}^N}}{\delta^p b^{n\,\delta^q}}$$

for $n \geqslant m-N$. From (a) and property (i) of g,

$$\|u-f\|_K \leqslant \|u-g\|_{K_{3\delta}} + \|g-f\|_K \leqslant k\omega(\delta).$$

Thus

$$d_n(f, K) \leqslant k\omega(\delta) + C\,\frac{k\omega(\delta) + \|f\|_{\mathbf{R}^N}}{\delta^p b^{n\delta^q}}$$

for $n \geqslant m-N$.

Since $b>1$, one can verify that there exist positive numbers v and ε depending only on b, p and q with the following property:

$$\frac{n^{pv}}{b^{n^{(1-qv)}}} \leqslant \frac{1}{n^\varepsilon} \qquad \text{for each positive integer } n \geqslant m-N.$$

Upon setting $\delta = 1/n^{\nu}$ and using the fact that

$$\omega(t) \geqslant \frac{\omega(1)}{2}\, t,$$

valid for the modulus of continuity ω of any non-constant continuous function f in $\mathbf{R}^N$, we complete the proof of Theorem 2.

5. Remarks

In this section, we assume that $m > N$ and N is even and prove a version of Theorem 2 in this setting. let $\omega(\delta)$ be a modulus of continuity, i.e., ω is a positive, increasing function of $\delta \geqslant 0$ such that $\omega(\delta_1 + \delta_2) \leqslant \omega(\delta_1) + \omega(\delta_2)$. We make the additional assumption that

$$\omega(\delta) \log \frac{1}{\delta} \leqslant \beta \omega(\delta^{\sigma})$$

for some positive constants β and σ. Define

$$C^{\omega}(\mathbf{R}^N) \equiv \{ f \text{ continuous on } \mathbf{R}^N \colon |f(x) - f(y)| \leqslant \omega(|x-y|) \text{ for all } x, y \in \mathbf{R}^N \}.$$

As an example, this hypothesis is satisfied when $\omega(\delta) = \delta^{\gamma}$ for some $\gamma \in (0, 1]$, which yields the classical space of functions satisfying a Hölder–Lipschitz condition of order γ. Let K be a compact set in $\mathbf{R}^N$ whose complement is a John domain. Then we have the following elliptic Jackson theorem: *there are constants C_1, $C_2 > 0$, depending only on ω, K and L such that for any $f \in C^{\omega}(\mathbf{R}^N)$ with compact support satisfying $L(D) f = 0$ on the interior of K, and for any positive integer $n \geqslant m - N$, (2) holds.*

Recall that $E(x) = E_1(x) + E_2(x) \log |x|$ in this setting. Since (23) remains valid if $m \geqslant N$ and N is even, the estimate on I_2 (equation (27)) is satisfied. Thus the only modification in the proof occurs in estimating I_1 (equation (26)). By modifying Lemma 8 to take into account logarithmic terms, we can prove: *if $m > N$ and N is even, there exists c_1 depending only on K and L such that for any $\delta \in (0, 1)$ and any $y \in \Omega_{\delta} \cap K_{3\delta}$,*

$$|\psi_y(x)| \leqslant c_1 (1 + |\log |x - \tilde{y}||) \qquad \text{for} \quad x \in K_{3\delta} \tag{22'}$$

and hence, using (28),

$$I_1 \leqslant \frac{k\omega(\delta)}{\delta^m} \left[\int_{\{|z| < c_3 \delta\}} |E(z)|\, dz + k(1 + |\log \delta|)\, \delta^{m-N} \operatorname{vol}(\mathbf{B}_{c_3\delta}(x)) \right].$$

Using the homogeneity of E_1 and E_2, an elementary computation shows that

$$\int_{\{|z|<c_3\delta\}} |E(z)|\,dz \leqslant \int_{\{|z|<c_3\delta\}} |E_1(z)|\,dz + \int_{\{|z|<c_3\delta\}} |E_2(z)|\,|\log|z||\,dz$$

$$\leqslant k\,\delta^m\left[1+\log\frac{1}{\delta}\right].$$

Hence $I_1 \leqslant k\omega(\delta)\log(1/\delta) \leqslant \beta k\omega(\delta^\sigma)$; thus u satisfies

(a') $\|u-g\|_{K_{3\delta}} \leqslant \beta k\omega(\delta^\sigma)$

and the rest of the proof proceeds as before.

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